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LYAPUNOV FAMILIES OF PERIODIC MOTIONS IN A REVERSIBLE SYSTEM†

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The problem of the existence of local one-parameter families of periodic motions (Lyapunov families) adjoining the position of equilibrium of reversible systems is investigated. In the most general situation, an analogue of the well-known Lyapunov theory is obtained. The bifurcation of the Lyapunov families when a pair of roots of the characteristic equation passes through zero is analysed. In particular, it is shown that, with this scenario, in the non-degenerate case the zero values of the roots are fatal for Lyapunov families. The effect of a "non-holonomic constraint" is investigated. Periodic motions, close to permanent rotations about a vertical, for heavy homogeneous ellipsoid on an absolutely rough plane, are analysed in an appendix. © 2000 Elsevier Science Ltd. All rights reserved.

1. LYAPUNOV FAMILIES OF PERIODIC MOTIONS

Consider the phase portrait for the mathematical pendulum $\ddot{x} + \sin x = 0$. The presence of a sign-definitive energy integral $\dot{x}^2/2 - \cos x = h$ (h = const) in the neighbourhood of the zero equilibrium position enables us to establish the existence in this neighbourhood of a one-parameter Lyapunov family of closed trajectories—periodic motions. Here, the family can be parametrized either by the energy constant h, or by the amplitude of periodic motion at the instant it intersects the x axis ($\dot{x}$ axis), or by the period of the motion. The sign-definiteness of the integral is established by the quadratic term in the expansion of the potential energy $-\cos x$, which corresponds to a pair of pure imaginary roots of the first approximation equation $\ddot{x} + x = 0$.

It is not difficult to see that the solutions of the Lyapunov family are symmetrical both about the x axis and about the $\dot{x}$ axis. Furthermore, the requirement concerning the presence of an integral is not necessary for a Lyapunov family to exist. When there is a pair of pure imaginary roots, the angle on the trajectory in a fairly close neighbourhood of equilibrium changes monotonically. This ensures double intersection by the trajectory of the x axis ($\dot{x}$ axis). Therefore, provided the phase portrait is symmetrical about the x axis or $\dot{x}$ axis, from this we quickly deduce the existence of a Lyapunov family. In particular, such a situation occurs in a reversible system with one degree of freedom.

$$\ddot{x} + \omega^2 x = F(x, \dot{x}), \quad \omega = \text{const}$$

with non-linear terms $F(x, \dot{x})$ in each of the cases: (a) $F(x, -\dot{x}) = F(x, \dot{x})$; (b) $F(-x, \dot{x}) = -F(x, \dot{x})$. In the first of them, the trajectories of the Lyapunov family are symmetrical about the x axis, and in the second they are symmetrical about the $\dot{x}$ axis; symmetry about both axes is now not guaranteed.

Thus, for a Lyapunov family to exist it is sufficient to require either the existence of an integral of the phase portrait symmetry to occur (the presence of a pair of pure imaginary roots is obligatory). Therefore, the theory of Lyapunov families of periodic motions is developed for systems allowing of the first integral (Lyapunov systems) and for systems possessing phase portrait symmetry (reversible systems).

2. THE LYAPUNOV–BRYUNO–DEVANEY THEOREM

Let us consider the problem of the existence of local periodic motions of a smooth reversible system

$$\begin{gathered}\dot{\mathbf{u}} = \mathbf{A}\mathbf{v} + \mathbf{U}(\mathbf{u}, \mathbf{v}), \quad \dot{\mathbf{v}} = \mathbf{B}\mathbf{u} + \mathbf{V}(\mathbf{u}, \mathbf{v}); \quad \mathbf{u} \in \mathbb{R}^l, \mathbf{v} \in \mathbb{R}^n \ (l \geqslant n) \\ \mathbf{U}(\mathbf{0},\mathbf{0}) = \mathbf{0}, \quad \mathbf{V}(\mathbf{0},\mathbf{0}) = \mathbf{0}, \quad \mathbf{U}(\mathbf{u},-\mathbf{v}) = -\mathbf{U}(\mathbf{u},\mathbf{v}), \quad \mathbf{V}(\mathbf{u},-\mathbf{v}) = \mathbf{V}(\mathbf{u},\mathbf{v})\end{gathered} \tag{2.1}$$

(where $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, and $\mathbf{U}$ and $\mathbf{V}$ are non-linear terms) with the fixed points set $\mathbf{M} = \{\mathbf{u}.\mathbf{v}: \mathbf{v} = \mathbf{0}\}$.

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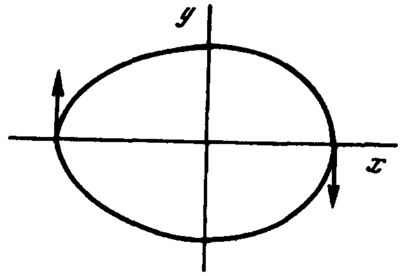

Fig. 1.

Theorem 1 (Lyapunov–Bryuno–Devaney). Suppose that: (a) the characteristic equation of the linear part of system (2.1) has a pair of $\pm i\omega$ pure imaginary roots; (b) among the other roots of this equation there is none equal to $\pm ip\omega$ ($p \in \mathbb{N}$); (c) rank $\mathbf{B} = n$. Then system (2.1) has an $(l-n)$-parameter manifold of equilibrium positions, belonging to the fixed points set $\mathbf{M}$ and containing zero equilibrium, and each point of this manifold has an adjoining one-parameter family of Lyapunov periodic motions.

This formulation refines the corresponding previous assertion in [1]. When investigating the case $l = n$ [2, 3], condition c was replaced with the stronger condition that there are no zero roots. Obviously, when $l = n$ there are no zero roots if rank $\mathbf{A}$ = rank $\mathbf{B} = n$. However, if rank $\mathbf{A} = n - k$, and rank $\mathbf{B} = n$, then there are k pairs of zero roots. However, the theorem remains valid in this case.

According to Theorem 1, when $l = n = 1$, a singular point in the (u, v) plane is the centre. When $l > n = 1$, the space (u, v) consists of an $(l-1)$-parameter manifold of equilibrium positions, each point of which has an adjoining Lyapunov family; there are no other local solutions.

The proof of Theorem 1 will be anticipated by a lemma.

Lemma 1. In the case when rank $\mathbf{B} = n$, system (2.1), by a non-singular linear transformation, can be reduce to the form

$$\boldsymbol{\xi}^{\boldsymbol{\cdot}} = \mathbf{P}\mathbf{y} + \boldsymbol{\Xi}(\boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad \mathbf{x}^{\boldsymbol{\cdot}} = \mathbf{J}\mathbf{y} + \mathbf{X}(\boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad \mathbf{y}^{\boldsymbol{\cdot}} = \mathbf{x} + \mathbf{Y}(\boldsymbol{\xi}, \mathbf{x}, \mathbf{y}),$$
$$\boldsymbol{\xi} \in \mathbf{R}^{l-n}, \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^n \tag{2.2}$$

(where $\boldsymbol{\Xi}$, $\mathbf{X}$, $\mathbf{Y}$ are non-linear terms) with a real Jordan matrix $\mathbf{J}$ with real eigenvalues $\lambda_1^2, \ldots, \lambda_n^2$ and the fixed points set $\mathbf{M}_* = \{\xi, \mathbf{x}, \mathbf{y}: \mathbf{y} = \mathbf{0}\}$.

Proof. We shall write the first approximation equations in (2.1) in the form

$$\dot{\mathbf{u}}_1 = \mathbf{A}_1\mathbf{v}, \ \dot{\mathbf{u}}_2 = \mathbf{A}_2\mathbf{v}, \ \mathbf{v}^{\boldsymbol{\cdot}} = \mathbf{B}_1\mathbf{u}_1 + \mathbf{B}_2\mathbf{u}_2$$
$$\mathbf{u}_1 \in \mathbf{R}^{l-n}, \ \mathbf{u}_2 \in \mathbf{R}^n, \ \det \mathbf{B}_2 \neq 0$$

and, instead of $\mathbf{u}_2$, select a new variable $\mathbf{B}_2\mathbf{u}_2 + \mathbf{B}_1\mathbf{u}_1$. Then $\dot{\mathbf{v}} = \mathbf{u}_2$. Further, we make the transformation

$$\mathbf{x} = \mathbf{C}^{-1}\mathbf{u}_2, \ \mathbf{y} = \mathbf{C}^{-1}\mathbf{v} \ (\det \mathbf{C} \neq 0)$$

We have

$$\dot{\mathbf{u}}_1 = \mathbf{A}_1\mathbf{C}\mathbf{y}, \ \mathbf{x}^{\boldsymbol{\cdot}} = \mathbf{C}^{-1}\mathbf{A}^2\mathbf{C}\mathbf{y}, \mathbf{y}^{\boldsymbol{\cdot}} = \mathbf{x}$$

It can be seen that the characteristic equation of this system has $l - n$ simple zero roots. The remaining roots $\pm\lambda$ are determined from a polynomial of order n in λ^2. Here, λ^2 are real numbers and they coincide with the eigenvalues $\lambda^2_1, \ldots, \lambda^2_n$ of the matrix $\mathbf{C}^{-1}\mathbf{A}_2\mathbf{C}$. The selection of the matrix $\mathbf{C}$ from the condition $\mathbf{C}^{-1}\mathbf{A}_2\mathbf{C} = \mathbf{J}$ completes the proof.

Proof of Theorem 1. By virtue of the oddness of the functions $\boldsymbol{\Xi}(\xi, \mathbf{x}, \mathbf{0})$, $\mathbf{X}(\xi, \mathbf{x}, \mathbf{y})$ with respect to the variable $\mathbf{y}$, we have $\boldsymbol{\Xi}(\xi, \mathbf{x}, \mathbf{0}) \equiv \mathbf{0}$, $\mathbf{X}(\xi, \mathbf{x}, \mathbf{0}) \equiv \mathbf{0}$. Therefore, system (2.2) admits of the constant solution $\xi = \xi^0$, $\mathbf{x} = \mathbf{x}^0(\xi^0)$, $\mathbf{y} = \mathbf{0}$, determined from the equation

$$\mathbf{x}^0 + \mathbf{Y}(\boldsymbol{\xi}^0, \mathbf{x}^0, \mathbf{0}) = \mathbf{0}$$

As a result, we have an $(l-n)$-parametric manifold of equilibrium positions (from ξ^0) belonging to fixed points set $\mathbf{M}^*$ and containing zero equilibrium $\xi = \mathbf{0}$, $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$.

Now, by making the replacement $(\xi, \mathbf{x}, \mathbf{y}) \to (\xi, \mu\mathbf{x}, \mu\mathbf{y})$, we reduce system (2.2) to the form

$$\begin{aligned} \xi^{\cdot} &= \mu\mathbf{P}\mathbf{y} + \mu\Sigma_*(\mu, \xi, \mathbf{x}, \mathbf{y}) \\ \mathbf{x}^{\cdot} &= \mathbf{J}\mathbf{y} + \mu\mathbf{X}_*(\mu, \xi, \mathbf{x}, \mathbf{y}) \\ \mathbf{y}^{\cdot} &= \mathbf{x} + \mu\mathbf{Y}_*(\mu, \xi, \mathbf{x}, \mathbf{y}) \end{aligned} \tag{2.3}$$

Hence, when $\mu = 0$, we obtain the linear generating system

$$\xi^{\cdot} = \mathbf{0}, \quad \mathbf{x}^{\cdot} = \mathbf{J}\mathbf{y}, \quad \mathbf{y}^{\cdot} = \mathbf{x} \tag{2.4}$$

From the final equation in (2.4) it can be seen that the zero root $\lambda = 0$ in the subsystem for $\mathbf{x}$, $\mathbf{y}$, if such exists, does not lead to a periodic solution. Therefore, when there is a pair of pure imaginary roots $\lambda_1 = \pm i\omega$ and the remaining roots $\lambda_s \neq \pm ip\omega$ $(p \in \mathbb{N})$, system (2.4) admits of a unique, from (ξ^*, a_1), family of $2\pi/\omega$-periodic solutions

$$\xi = \xi^*, \; x_1 = a_1\,\omega\cos\omega t, \; y_1 = a_1\sin\omega t, \; x_2 = \ldots = x_n = y_2 = \ldots = y_n = 0 \tag{2.5}$$

As $a_1 \to 0$, we have $(\xi, \mathbf{x}, \mathbf{y}) \to (\xi^*, \mathbf{0}, \mathbf{0})$, and solution (2.5) is shrinked to a point belonging to the manifold of equilibrium positions. Therefore, family (2.5) depends on the single important parameter a_1 and adjoins the equilibrium $(\xi^*, \mathbf{0}, \mathbf{0})$.

On an arbitrary symmetrical solution of system (2.4) we have

$$y_1 = a_1\sin\omega t, \quad y_s = \sum_{j=2}^{n} a_j\psi_{sj}(t), \quad a_j = \text{const}, \quad \psi_{sj}(-t) = -\psi_{sj}(t), \quad s, j = 2, \ldots, n$$

and here $\det \|\psi_{sj}(\pi/\omega)\| \neq 0$ by virtue of the uniqueness of family (2.5). Therefore, by the theorem of continuation with respect to the parameter of the symmetrical periodic solution of an autonomous reversible system [1, 4], we deduce the existence in (2.3), for sufficiently small $\mu \neq 0$, of a family, from (ξ^*, a_1), of symmetrical $T(\mu)$-periodic solutions, $T(0) = 2\pi/\omega$, originating from solutions (2.5). Then, taking into account the form of the transformation of system (2.2) to system (2.3), we establish, for a fixed value of a_1, for example for $a_1 = 1$, the existence in (2.2) of a family, from μ, of symmetrical $T(\mu)$-periodic solutions

$$\begin{aligned} &\xi = \xi^* + o(\mu), \; x_1 = \mu\omega\cos\omega t + o(\mu), \; y_1 = \mu\sin\omega t + o(\mu) \\ &x_2 = o(\mu), \ldots, x_n = o(\mu), \; y_2 = o(\mu), \ldots, y_n = o(\mu) \end{aligned}$$

adjoining the equilibrium $(\xi^*, \mathbf{0}, \mathbf{0})$.

The proof of Theorem 1 is complete.

Example 1. The equations of motion of the three-body problem [5]

$$\begin{aligned} &x^{\cdot\cdot} - 2y^{\cdot} = \frac{\partial W}{\partial x}, \quad y^{\cdot\cdot} + 2x^{\cdot} = \frac{\partial W}{\partial y} \\ &W = \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \quad r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x+\mu-1)^2 + y^2 \end{aligned} \tag{2.6}$$

describe a conservative system with two degrees of freedom and at the same time are reversible with the fixed points set $\{x, y, \dot{x}, \dot{y} : y = 0, \dot{x} = 0\}$. The collinear libration points—the constant solutions of system (2.6)—belong to a fixed point set; on these solutions $y = \dot{x} = \dot{y} = 0$. These points are unstable. However, the characteristic equation has a pair of pure imaginary roots [5]. Both from Lyapunov's theorem [6] and from Theorem 1 it follows that, in the neighbourhood of the collinear libration points adjoining them, one-parameter families of symmetrical periodic orbits exist. These orbits in the neighbourhood of one of the libration points were first constructed by Euler [7]. Here, Lyapunov families were found for the first time in one of the fundamental problems of mechanics.

Example 2. The Euler–Poisson equations of motion of a heavy rigid body with one fixed point [8]

$$A\frac{dp}{dt}+(C-B)qr=P(z_0\gamma_2-y_0\gamma_3),\quad \frac{d\gamma_1}{dt}+(q\gamma_3-r\gamma_2)=0 \qquad (2.7)$$
$$(A, B, C),\ (p, q, r),\ (\gamma_1, \gamma_2, \gamma_3)$$

are reversible with a fixed points set $\mathbf{M} = \{p, q, r, \gamma_1, \gamma_2, \gamma_3: p = q = r = 0\}$. When $y_0 = 0$, these equations have one further fixed points set on which $q = 0, \gamma_2 = 0$. If $x_0 = L\cos\alpha_1$ and $z_0 = L\cos\alpha_3$, then, in the lower equilibrium position, we have $p = q = r = 0$, $\gamma_1 = -\cos\alpha_1$ and $\gamma_3 = -\cos\alpha_3$. The equations of perturbed motion in the vicinity of the equilibrium in question have the form (2.1), where $l = 4$ and $n = 2$. Consequently, the characteristic equation has a pair of simple zero roots. The remaining roots λ are determined from the equation

$$\varkappa^2+\left(\frac{PL}{A}\cos^2\alpha_3+\frac{PL}{B}+\frac{PL}{C}\cos^2\alpha_1\right)\varkappa+\frac{P^2L^2}{AB}\cos^2\alpha_3+\frac{P^2L^2}{BC}\cos^2\alpha_1=0$$

where $\varkappa = \lambda^2$. It can be seen that this equation, with $L \neq 0$, has a pair of negative roots. This means that the characteristic equation has two pairs of pure imaginary roots. Furthermore, in equations of type (2.1) we have

$$\mathbf{B}=\begin{Vmatrix} -Pz_0/B & Px_0/B & 0 & 0 \\ 0 & 0 & \cos\alpha_3 & \cos\alpha_1 \end{Vmatrix}$$

and, for $L \neq 0$, we derive rank $\mathbf{B} = 2$. Therefore (Theorem 1), in the vicinity of the lower equilibrium position in the non-resonance case there are two families of small vibrations. In the resonance case, at least one of these families is retained.

Furthermore, it follows from Theorem 1 that a double-parameter family of constant solutions belonging to a stationary set exists. The existence of this family is also obvious from the initial equations (2.7). In this family we have

$$(A-C)\,rp + P(z_0\gamma_1-x_0\gamma_3) = 0,\quad r\gamma_1-p\gamma_3 = 0$$

Since the geometric integral contains no arbitrary constant, then, unlike the general situation, this is a single-parameter family. For each point of this family there is a corresponding rotation of the rigid body with a constant angular velocity, to which Lyapunov families of periodic motions adjoin.

Note that, formally, Lyapunov's theorem [6] is not applicable here ($l > n$). However, using integrals (the geometric integral and the integral of the angular momentum), it is possible to reduce the problem to a Lyapunov system. In the following problem, the Lyapunov theorem is essentially inapplicable.

Example 3. Let us consider a discrete model of an elastic rod loaded with a tracking force ([9, p.105]). The mechanical system lies in a horizontal plane and consists of two identical rods of mass m and length l, connected to each other and to the stationary base by ideal joints and springs of stiffness c_1 and c_2 respectively. The free end of the second rod is acted upon by a constant tracking force F directed along the axis of the rod. A rectilinear configuration of the system corresponds to the undeformed state of the springs.

We have a mechanical system with positional forces. This system is reversible [10]. If the positions of the rods are determined by the angles of their deviation from the equilibrium state, then the characteristic equation of the linearized equations of perturbed motion has the form

$$7\lambda^4 + 6a/(ml^2)\lambda^2 + 36c_1c_2/(m^2l^4) = 0,\quad a = 2c_1 + 16c_2-5Fl$$

From Theorem 1 it follows that, when the following inequality is satisfied

$$a-2\sqrt{7c_1c_2} > 0$$

there is always one Lyapunov family of periodic motions. When the condition of non-resonance is satisfied we have two families of this kind. Obviously, these families describe small fluctuations of the rods about the equilibrium position.

3. THE CASE OF rank $\mathbf{B} = n - 1$

We will investigate the problem of the existence of Lyapunov families of periodic motions when one of the conditions in Theorem 1 is not satisfied. The case when in a system with $l = n$, there is one further pair of pure imaginary roots of the form $\pm ip\omega (p \in \mathbb{N})$ was considered in [11, 12]. Here it was assumed that rank $\mathbf{B} = n$.

Now, suppose rank $\mathbf{B} = n - 1$ in system (2.1). Then, when the condition rank $\mathbf{A} = n$ is satisfied, system (2.1), by a non-singular linear transformation, can be reduced to the form

$$\begin{aligned}
&\boldsymbol{\xi}^{\bullet} = \boldsymbol{\Sigma}\,(\boldsymbol{\xi}, x, y, \mathbf{p}, \mathbf{q}), \quad \boldsymbol{\xi} \in \mathbb{R}^{l-n} \\
&x^{\bullet} = y, \quad y^{\bullet} = Y_0(x) + Y_1(\boldsymbol{\xi}, x, y, \mathbf{p}, \mathbf{q}) \\
&\mathbf{p}^{\bullet} = \mathbf{A}_*\mathbf{q} + \mathbf{P}(\boldsymbol{\xi}, x, y, p, \mathbf{q}) \\
&\mathbf{q}^{\bullet} = \mathbf{B}_*\mathbf{p} + \mathbf{Q}(\boldsymbol{\xi}, x, y, \mathbf{p}, \mathbf{q}); \quad \mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-1}
\end{aligned} \tag{3.1}$$

where $\mathbf{A}_*$ and $\mathbf{B}_*$ are constant square $(n-1)$-matrices, and Σ, Y_0, Y_1, $\mathbf{P}$ and $\mathbf{Q}$ are non-linear terms. System (3.1) is reversible and has the fixed points set $\mathbf{M}_* = \{\xi, x, y, \mathbf{p}, \mathbf{q}: y = 0, \mathbf{q} = \mathbf{0}\}$.

When

$$Y_1\,(\boldsymbol{\xi}, 0, 0, \mathbf{p}, \mathbf{q}) \equiv 0 \tag{3.2}$$

the space $(\xi, \mathbf{p}, \mathbf{q})$ is invariant for system (3.1). In this space the motion is described by the system

$$\begin{aligned}
&\boldsymbol{\xi}^{\bullet} = \boldsymbol{\Xi}\,(\boldsymbol{\xi}, 0, 0, \mathbf{p}, \mathbf{q}) \\
&\mathbf{p}^{\bullet} = \mathbf{A}_*\mathbf{q} + \mathbf{P}(\boldsymbol{\xi}, 0, 0, \mathbf{p}, \mathbf{q}) \\
&\mathbf{q}^{\bullet} = \mathbf{B}_*\mathbf{p} + \mathbf{Q}(\boldsymbol{\xi}, 0, 0, \mathbf{p}, \mathbf{q})
\end{aligned} \tag{3.3}$$

for which Theorem 1 holds. Therefore, when condition b of this theorem is satisfied, for each pair of pure imaginary roots of the characteristic equation there is a corresponding Lyapunov family of periodic motions.

Let us now assume that condition (3.2) is not satisfied Then, in the non-singular case we have

$$Y_1(\mathbf{0}, 0, 0, \mathbf{p}, \mathbf{0}) = \sum_{s,j=1}^{n-1} c_{sj} p_s p_j + o(\|\mathbf{p}\|^2), \quad c_{sj} = \text{const} \tag{3.4}$$

We shall introduce a small parameter μ by the replacement $(\xi, x, y, \mathbf{p}, \mathbf{q}) \to (\mu\xi, \mu x, \mu y, \mu\mathbf{p}, \mu\mathbf{q})$. Then, on a Lyapunov family caused by a pair of pure imaginary roots $\pm i\omega$ and adjoining the zero position of equilibrium, if such a family exists, we shall obtain

$$p_s = a_s \cos \omega t + \ldots, \quad q_s = b_s \sin \omega t + \ldots \quad (a_s, b_s = \text{const})$$

In this case $\dot{\xi}, y\,\ddot{x}$ are at least of the first-order in μ. Hence, in periodic motion, the variables ξ, and x are of the same order in μ, and the rate of change of the variable y on intersecting the fixed set $\mathbf{M}_*$ is equal to

$$y^{\bullet} = \mu \sum_{s,j=1}^{n-1} c_{sj} a_s a_j + o(\mu)$$

and has the same sign. Obviously, this is impossible in periodic motion (Fig. 1).

Theorem 2. Suppose that in system (2.1) we have rank $\mathbf{A} = n$ and rank $\mathbf{B} = n - 1$. Then, in system (2.1) there is no Lyapunov family of periodic motions that adjoins the zero position of equilibrium.

Note. The condition rank $\mathbf{A} = n$ guarantees that system (2.1) can be reduced to the form (3.1), and here Theorem 2 holds for system (3.1) irrespective of whether the condition $A_* = n - 1$ is satisfied. Therefore, the conditions rank $\mathbf{A} = n$ and $\mathbf{B} = n - 1$ in Theorem 2 can be replaced by the condition that a pair of zero roots is present with a Jordan cell and variable y equal to zero on a fixed set.

Thus, in the general situation of (3.4), non-satisfaction of condition c in Theorem 1, as opposed to condition b [11, 12], is fatal for a Lyapunov family. If system (2.1) contains a small parameter ε, has a pair of pure imaginary roots $\pm i\omega(\varepsilon)$, $\omega(0) \neq 0$ and, when $\varepsilon = 0$, two other roots pass through zero by the scenario of non-satisfaction of condition c, then, when $\varepsilon = 0$, the Lyapunov family disappears.

An interesting problem arises concerning periodic motions that are caused by a pair of zero roots and close to plane periodic motions of the system

$$x^{\bullet} = y, \quad y^{\bullet} = Y_0(x) + Y_1(\mathbf{0}, x, y, \mathbf{0}, \mathbf{0}) \tag{3.5}$$

We shall assume that

$$Y_0(x) = gx^m + o(x^m), \quad g = \text{const}$$

Then, the function $V = xy$ is the Chetayev function for system (3.5) in the case of even m, and, in the case of odd m, when $g > 0$ [13]. From this it follows that, in the given cases, all solutions of system (3.5) from the region $xy > 0$ are departing, and therefore there are no periodic solutions. In particular, such a situation occurs in the most general case when $m = 2$.

Suppose that $m = 3$. In spite of the degeneracy, this case generally occurs in mechanical problems. Assuming that $g < 0$, w e change to the variables r, θ according to the formulae

$$x = r\cos\theta, \quad y = r^2 \sin\theta$$

As a result, we obtain a system that is 2π-periodic in θ

$$\begin{aligned} r^{\cdot} &= r^2 \frac{(1+g\cos^2\theta)}{1+\sin^2\theta}\sin\theta\cos\theta + r^3 R(r,\theta) \\ \theta^{\cdot} &= -r\frac{2\sin^2\theta - g\cos^4\theta}{1+\cos^2\theta} + r^2\Theta(r,\theta) \end{aligned} \tag{3.6}$$

From the second equation it can be seen that, for sufficiently small r, the angle θ varies monotonically along the trajectories. Therefore, the angle θ can be chosen as a new independent variable, and system (3.6) can be written in the form of a single reversible equation for r, describing the motion on a fixed points set. It is well known [14] that all solutions of the equation obtained will be 2π-periodic in θ. Hence, when $m = 3$ and $g < 0$, the zero of system (3.5) will be the centre.

In the presence of an associated system (for the variables ξ, $\mathbf{p}$, $\mathbf{q}$), it was not possible to prove the existence of a family of periodic motions adjoining zero.

4. BIFURCATION OF LYAPUNOV FAMILIES OF PERIODIC MOTIONS

We shall assume that system (2.1) contains a parameter ε, and when $\varepsilon \neq 0$ we have rank $\mathbf{B}(\varepsilon) = n$ and rank $\mathbf{B}(0) = n - 1$. Then, when the condition rank $\mathbf{A}(0) = n$ is satisfied, system (2.1) with $\varepsilon = 0$ reduces to the form (3.1). From (3.1) we also derive a transformed system when $\varepsilon \neq 0$ if ε is regarded as a local variable corresponding to the equation $\dot{\varepsilon} = 0$. Then, introducing, where necessary, a new parameter—a function of ε—we have the system

$$\begin{aligned} &\xi^{\cdot} = \Xi(\varepsilon,\xi,x,y,\mathbf{p},\mathbf{q}), \quad \xi \in \mathbb{R}^{l-n} \\ &x^{\cdot} = y, \quad y^{\cdot} = \varepsilon x + Y_0(\varepsilon,x) + Y_1(\varepsilon,\xi,x,y,\mathbf{p},\mathbf{q}) \\ &\mathbf{p}^{\cdot} = \mathbf{A}_*(\varepsilon)\mathbf{q} + \mathbf{P}(\varepsilon,\xi,x,y,\mathbf{p},\mathbf{q}) \\ &\mathbf{q}^{\cdot} = \mathbf{B}_*(\varepsilon)\mathbf{p} + \mathbf{Q}(\varepsilon,\xi,x,y,\mathbf{p},\mathbf{q}); \quad \mathbf{p},\mathbf{q} \in \mathbb{R}^{n-1} \end{aligned} \tag{4.1}$$

where $\mathbf{A}_*(\varepsilon)$ and $\mathbf{B}_*(\varepsilon)$ are constant matrices, det $\mathbf{B}_* \neq 0$, and the non-linear functions Ξ, Y_1, $\mathbf{P}$ and $\mathbf{Q}$ vanish at $\xi = 0, x = y = 0, \mathbf{p} = \mathbf{q} = \mathbf{0}$. Furthermore, $Y_1(\varepsilon, \mathbf{0}, x, \mathbf{0}, \mathbf{0}) \equiv 0$.

Lemma 2. Suppose that

$$Y_0(\varepsilon,x) = g(\varepsilon)x^m + o(x^m), \quad g(0) \neq 0$$

Then, when $m = 2$, system (4.1) admits of two $(l - n)$-parametric families of equilibrium positions belonging to a fixed set. When $m = 3$ and $\varepsilon g(0) < 0$ there are three such families, but when $m = 3$ and $\varepsilon g(0) > 0$ there is only one such family. In all of these cases, one of the families contains a zero equilibrium.

Proof. Since on the fixed points set $\mathbf{M}_*$, the functions Ξ and $\mathbf{P}$ vanish, the problem reduces to the compatibility of the system of equations

$$\varepsilon x + Y_0(\varepsilon, x) + Y_1(\varepsilon, \boldsymbol{\xi}, x, 0, \mathbf{p}, \mathbf{0}) = 0$$
$$\mathbf{B}_*(\varepsilon)\mathbf{p} + \mathbf{Q}(\varepsilon, \boldsymbol{\xi}, x, 0, \mathbf{p}, \mathbf{0}) = \mathbf{0}, \ \det\mathbf{B}_*(\varepsilon) \neq 0 \qquad (4.2)$$

When $\boldsymbol{\xi} = \mathbf{0}$, from the second equation of system (4.2) we determine $\mathbf{p}(\varepsilon, x)$ as a non-linear function of x, and here $\mathbf{p}(\varepsilon, 0) = \mathbf{0}$. We substitute into the first equation.
We obtain the equation

$$f(\varepsilon, x) \equiv \varepsilon x + g(0)x^m + o(x^m) + x^m o(\varepsilon) = 0 \quad (m = 2, 3)$$

which has the obvious root $x^0 = 0$, and also the other roots in Lemma 2. Here, when any of these roots are substituted, the partial derivative $\partial f/\partial x \neq 0$. Therefore, it follows from the implicit function theorem that, for sufficiently small $\boldsymbol{\xi} = \boldsymbol{\xi}^0$, system (4.2) has an $(l-n)$-parametric family (from $\boldsymbol{\xi}^0$) of roots $x^0(\boldsymbol{\xi}^0)$ and $x^*(\boldsymbol{\xi}^0)$.

Note. It follows from the proof that $\boldsymbol{\xi}^0$ is of the order of $o(x^0)$ when $m = 2$, and $o[(x^0)^{3/2}]$ when $m = 3$.

In the (x, y) plane these families of equilibrium positions are represented by points. The nature of these singular points is shown in Fig. 2 ($m = 2$) and Fig. 3 ($m = 3$). Here, from left to right we show the cases $\varepsilon < 0$, $\varepsilon = 0$ and $\varepsilon > 0$ respectively, and in the upper (lower) diagrams in Figs. 2 and 3 we have $g > 0$ ($g < 0$). Note that Fig. 3 also arises [15] when studying a Hamiltonian system.

Theorem 3. Let the matrices $\mathbf{A}_*(0)$ and $\mathbf{B}_*(0)$ in system (4.1) satisfy the conditions of theorem 1 imposed on the matrices **A** and **B** respectively. Then, for $\varepsilon \neq 0$, each point of the $(l-n)$-manifold of equilibrium positions shown in Figs 2 and 3 has adjoining Lyapunov families of periodic motions, and here the

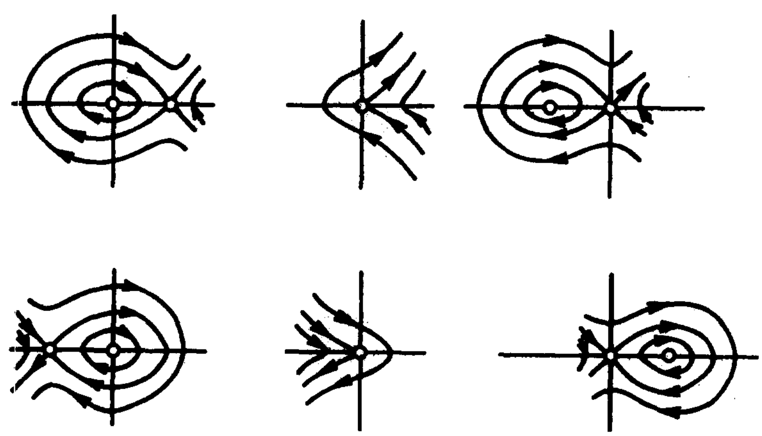

Fig. 2.

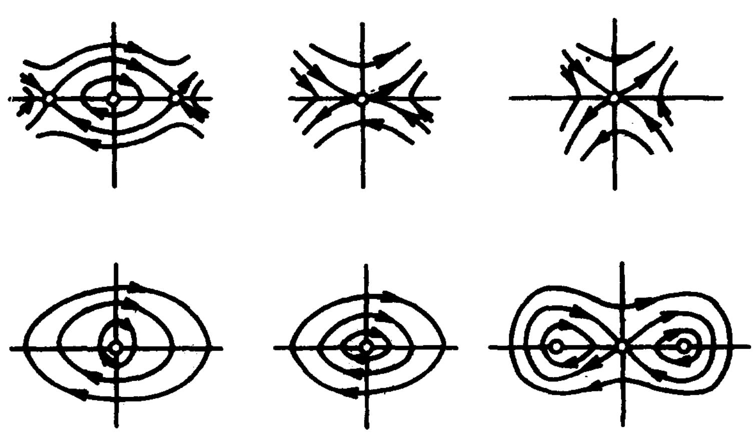

Fig. 3.

families for "saddle"-type equilibria are orbitally unstable. Furthermore, Lyapunov families of periodic motions that are similar to those shown in Figs 2 and 3, with a frequency close to the number $k\sqrt{|\varepsilon|}$ (k = const), adjoin the "centre"-type equilibrium positions.

Proof. If the matrices $\mathbf{A}_*(0)$ and $\mathbf{B}_*(0)$ satisfy the conditions of Theorem 1 imposed on the matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, then, for fixed $\varepsilon \neq 0$, the system of linear approximation in (4.1) satisfies all the conditions of Theorem 1. This indicates the correctness of the first assertion of Theorem 3. The instability of the Lyapunov motions adjoining "saddle"-type equilibria follows from the instability, in the first approximation, for the variables x and y. Here, the departure of the solutions occurs exponentially with exponent $k\sqrt{|\varepsilon|}$ (k = const). The existence of yet another Lyapunov family adjoining a "centre"-type equilibrium can also be deduced from Theorem 1. This family corresponds to a pair of pure imaginary roots $\pm ik\sqrt{|\varepsilon|}$, for which, for sufficiently small ε, condition b of Theorem 1 is satisfied.

From Figs 2 and 3, taking Theorems 2 and 3 into account, the scenario of bifurcations of Lyapunov families of periodic motions when a pair of roots passes through zero becomes clear.

5. THE EFFECT OF A "HOLONOMIC" CONSTRAINT

The above investigation was carried out for arbitrary l and n satisfying the condition $l \cap n$. Otherwise, Theorems 1–3 contain assertions that hold when $l = n$ and can be generalized to the case when $l > n$. However, when solving applied problems, in particular in non-holonomic mechanics, we always have $l \cap n$. Therefore, the question of the "active contribution" of the variable ξ to Lyapunov families of periodic motions is of interest.

We will examine system (3.1) and, initially assuming that $m = 2$, we will single out in the function Y_1 the terms quadratic in x and ξ

$$Y_*(\xi, x) \equiv gx^2 + x\Sigma\alpha_s\xi_s + \Sigma\beta_{sj}\xi_s\xi_j \quad (\alpha_s, \beta_{sj} = \text{const})$$

All equilibrium positions of the system that belong to a fixed points set are determined from the system of non-linear equations

$$\begin{aligned} &gx^2 + x\Sigma\alpha_s\xi_s + \Sigma\beta_{sj}\xi_s\xi_j + Y_{**}(\xi, x, 0, \mathbf{p}, 0) = 0 \\ &\mathbf{B}_*\mathbf{p} + \mathbf{Q}(\xi, x, 0, \mathbf{p}, 0) = 0 \end{aligned} \tag{5.1}$$

where $Y_{**}(\xi, x, \mathbf{p}, \mathbf{q})$ is the right-hand side of the equation for y without the function $Y_*(\xi, x)$.

From the second equation of system (5.1) we determine the non-linear function $\mathbf{p}(\xi, x)$ and substitute it into the first equation. Then, after the introduction, by making the replacement $(\xi, x) \to (\mu\xi, \mu x)$, of the small parameter μ, we obtain an equation for determining x

$$gx^2 + x\Sigma\alpha_s\xi_s + \Sigma\beta_{sj}\xi_s\xi_j + \mu F(\mu, \xi, x) = 0 \tag{5.2}$$

When $\mu = 0$, Eq. (5.2), with the appropriate choice of the quantities $\xi_1, \ldots, \xi_{l-n}$, always has simple roots, provided all the constants α_s and β_{sj} are not simultaneously equal to zero or the conditions $\alpha_s = 0$ ($s = 1, \ldots, l-n$) are not satisfied and the quadratic form of ξ is not sign-definite with the sign of the number g. In this case, if all $\beta_{sj} = 0$, then one of the roots is zero. Consequently, with the exception of the given cases, system (5.1) allows of an $(l-n)$-parametric manifold (from ξ) of equilibrium positions that belong to the fixed points set. Here, the region of permissible values of ξ is specified by the condition for Eq. (5.2) to be solvable.

Now let us examine one of the equilibria of the manifold in question and change to its vicinity. Then, by virtue of the fact that the equilibrium belongs to the fixed points set, we obtain a reversible autonomous system. Theorem 1 can be applied to this system. As a result, we obtain the Lyapunov families of periodic motions that were established in Theorem 3.

A similar situation occurs for $m = 3$. Only in this case the function $Y_*(\xi, x)$ has the form

$$Y_*(\xi, x) = gx^3 + x\Sigma\alpha_s\xi_s$$

and the equation of equilibria (5.1) has one or three families of solutions provided not all α_s are simultaneously equal to zero. When $\alpha_s = 0$ ($s = 1, \ldots, l-n$), it is necessary to select

$$Y_*(\xi, x) = gx^3 + \Sigma\beta_{sj}\xi_s\xi_j$$

Note that, both for $m = 2$ and $m = 3$, it is possible to write new expressions for the function $Y_*(\xi, x)$ if all α_s and β_{sj} are equal to zero.

Theorem 4. Each point of the manifold of equilibrium positions of system (3.1), with the exception of zero equilibrium, defined by the simple root of the equation $Y_*(\xi, x) = 0$, has an adjoining Lyapunov family corresponding to a pair of pure imaginary roots of the subsystem for **p**, **q**. Here, for the matrices $\mathbf{A}_*$, $\mathbf{B}_*$, the conditions of Theorem 1, formulated for the matrices **A** and **B** respectively, should be satisfied. Families adjoining a "saddle"-type equilibrium are orbitally unstable, and "centre"-type equilibria have one further adjoining Lyapunov family similar to plane motions on (x, y).

Note. It follows from Theorem 4 that the effect of a "non-holonomic constraint" when condition c in Theorem 1 is not satisfied, i.e. when there is a pair of zero roots with a Jordan cell with the condition rank $\mathbf{B} = n - 1$, consists of the "distancing" of these roots from zero. Lyapunov families behave in the same way as in the case of a pair of roots close to zero with a Jordan cell.

6. A HEAVY, HOMOGENEOUS ELLIPSOID ON AN ABSOLUTELY ROUGH PLANE. PERIODIC MOTIONS CLOSE TO PERMANENT ROTATIONS ABOUT THE VERTICAL

The dynamics of a heavy, homogeneous ellipsoid on an absolutely rough horizontal plane is described by the equations

$$\begin{aligned}
&x^{\cdot} = \frac{a^2}{b^2}y\omega_3 - \frac{a^2}{c^2}z\omega_2 + \frac{a^2-c^2}{a^2c^2}x^2z\omega_2 + \frac{b^2-a^2}{b^2a^2}x^2y\omega_3 + \frac{c^2-b^2}{c^2b^2}xyz\omega_1 \equiv X \\
&[A + m(y^2+z^2)]\omega_1^{\cdot} - mxy\omega_2^{\cdot} - mxz\omega_3^{\cdot} = \\
&= (B-C)\omega_2\omega_3 + (X - y\omega_3 + z\omega_2)(\omega_1 x + \omega_2 y + \omega_3 z) - \\
&-m\omega_1(xx^{\cdot} + yy^{\cdot} + zz^{\cdot}) - mg\frac{c^2-b^2}{c^2b^2}xyz\delta \\
&\delta = (x^2/a^4 + y^2/b^4 + z^2/c^4)^{-1/2}, \quad A = m(b^2+c^2)/5, \quad B = m(c^2+a^2)/5, \\
&C = m(a^2+b^2)/5 \\
&(x, y, z), \quad (\omega_1, \omega_2, \omega_3), \quad (A, B, C), \quad (a, b, c)
\end{aligned} \tag{6.1}$$

where m is the mass of the ellipsoid, a, b and c are its semi-axes, A, B and C are the principal central moments of inertia, x, y, z are the coordinates of the point of contact of the ellipsoid and plane in a moving system of coordinates with the axes directed along the axes of the ellipsoid, ω_1, ω_2 and ω_3 are projections of the instantaneous angular velocity vector in the same system and δ is the distance from the centre of masses to the point of contact.

System (6.1) is reversible with three fixed points sets, on each of which one of the pairs of variables (x, ω_1), (y, ω_2), (z, ω_3) is equal to zero. Furthermore, as always for a heavy, rigid body on a rough plane, there is the fixed points set $\{x, y, z, \omega_1, \omega_2, \omega_3: \omega_1 = 0, \omega_2 = 0, \omega_3 = 0\}$.

We will change to a dimensionless form of the equations, introducing new variables, parameters and the time τ by the formulae

$$\begin{aligned}
&x = ax_1, \quad y = by_1, \quad z = cz_1, \quad \omega_1 = \omega p, \quad \omega_2 = \omega q, \quad \omega_3 = \omega r, \quad \tau = \omega t \\
&\alpha = a/c, \quad \beta = b/c, \quad \gamma = g/(\omega^2 c), \quad A_1 = (\beta^2+1)/5, \quad B_1 = (1+\alpha^2)/5, \\
&C_1 = (\alpha^2 + \beta^2)/5
\end{aligned}$$

where ω is a certain constant having the dimension of angular velocity. Then the system obtained has the particular solution

$$x_1 = y_1 = 0, \quad z_1 = -1, \quad p = q = 0, \quad r = \omega_0(\text{const}) \tag{6.2}$$

corresponding to permanent rotation abut the vertical with angular velocity $\omega\omega_0$. Assuming $\omega_0 \neq 0$, it is always sufficient to examine the case $\omega_0 = 1$. For this, on changing to a new time τ, we shall assume an angular velocity in permanent rotation equal to ω.

System (6.1) allows of two integrals—the energy integral and the geometric integral. In dimensionless form, these integrals have the form

$$(\beta y_1 r - z_1 q)^2 + (z_1 p - \alpha x_1 r)^2 + (\alpha x_1 q - \beta y_1 p)^2 + A_1 p^2 + B_1 q^2 + C_1 r^2 + \\ +2\gamma\left(\frac{x_1^2}{\alpha^2} + \frac{y_1^2}{\beta^2} + z_1^2\right)^{-1/2} = 2\frac{h}{m\omega^2 c^2} = 2h_1 (\text{const}) \tag{6.3}$$

where h is the energy constant.

We will compose the equations of perturbed motion in the vicinity of the particular solution (6.2). Then, in the reversible system of the form (2.1) obtained, we have $l = 4$ and $n = 2$. Consequently, there is a two-parameter family of equilibrium positions that belongs to the fixed points set. However, in the problem in question, the geometric integral contains no arbitrary constant. Therefore, the dimensionality of the manifold of equilibria is equal to unity.

We will eliminate the variables z_1 and r by means of integrals (6.3) and examine only isoenergetic motions in which the value of the energy integral is equal to its value on permanent rotation. As a result, to describe such motions in the variables x_1, y_1, p and q, we obtain a reversible system of the form (2.1) with $l = n = 2$ and matrices

$$\mathbf{A} = \begin{Vmatrix} \alpha/\beta & \alpha \\ \dfrac{5[\beta^2 - \alpha^2 + \gamma(\beta^2 - 1)]}{\beta(6+\beta^2)} & \dfrac{6 - 5\alpha^2 - \beta^2}{6+\beta^2} \end{Vmatrix}$$

$$\mathbf{B} = \begin{Vmatrix} -\beta/\alpha & -\beta \\ -\dfrac{5[\alpha^2 - \beta^2 + \gamma(\alpha^2 - 1)]}{\alpha(6+\alpha^2)} & -\dfrac{6 - \alpha^2 - 5\beta^2}{6+\alpha^2} \end{Vmatrix}$$

where, in the general case, rank $\mathbf{A}$ = rank $\mathbf{B}$ = 2.

The characteristic equation

$$\xi \geqslant 2, \ \lambda'(1) = 0$$

$$S\lambda^4 + G\lambda^2 + F = 0$$

$$S = (6+\alpha^2)(6+\beta^2)$$

$$G = S + 36(1-\alpha^2)(1-\beta^2) - 5\gamma[(6+\alpha^2)(1-\beta^2) + (6+\beta^2)(1-\alpha^2)]$$

$$F = (1-\alpha^2)(1-\beta^2)(6+5\gamma)^2$$

has two pairs of pure imaginary roots

$$\lambda_1 = \pm\left[\frac{-G - \sqrt{G^2 - 4SF}}{2S}\right]^{1/2}, \quad \lambda_2 = \pm\left[\frac{-G + \sqrt{G^2 - 4SF}}{2S}\right]^{1/2}$$

if

$$G > 2\sqrt{SF} \tag{6.4}$$

Condition (6.4) is satisfied identically if the rotation occurs about the least axis ($\alpha > 1$, $\beta > 1$), and, in the case of rotation about the greatest axis, if the angular velocity is sufficiently great [16]

$$\gamma < \frac{1}{5}\left[\frac{\sqrt{(6+\alpha^2)(6+\beta^2)} - 6\sqrt{(1-\alpha^2)(1-\beta^2)}}{\sqrt{(6+\alpha^2)(1-\beta^2)} + \sqrt{(6+\beta^2)(1-\alpha^2)}}\right]^2$$

It follows from Theorem 1 that in these cases there is always a single Lyapunov family of periodic motions, adjoining permanent rotation and corresponding to a pair of roots λ_1. When $\lambda_1^2 \neq k^2\lambda_2^2$ ($k \in \mathbb{N}$), again there is a second family corresponding to a pair of roots λ_2.

When rotation occurs about the middle axis ($\alpha < 1$, $\beta > 1$) we have $F \cup 0$, and the characteristic equation contains a pair of pure imaginary roots λ_1. A family of Lyapunov periodic motions that adjoins permanent rotation again corresponds to these roots (Theorem 1).

We will now assume that the ellipsoid of rotation performs permanent rotations about the axis of the ellipsoid, which is not an axis of symmetry ($\alpha = 1$). In this case, the matrices **A** and **B** will take the form

$$\mathbf{A} = \left\| \begin{matrix} 1/\beta & 1 \\ \dfrac{5(\beta^2-1)(1+\gamma)}{\beta(6+\beta)^2} & \dfrac{1-\beta^2}{6+\beta^2} \end{matrix} \right\|, \quad \mathbf{B} = \left\| \begin{matrix} -\beta & -\beta \\ \dfrac{5(\beta^2-1)}{7} & \dfrac{5(\beta^2-1)}{7} \end{matrix} \right\|$$

and, when $\beta \neq 0$ and $\beta \neq 1$, we obtain rank $\mathbf{A} = 2$ and rank $\mathbf{B} = 1$. Hence, the characteristic equation has a pair of zero roots with one group of solutions and a pair of pure imaginary roots $\pm i\omega_*$ [16]

$$\omega_*^2 = 1 - 5\gamma(1-\beta^2)/(6+\beta^2)$$

if

$$5\gamma(1-\beta^2) < (6+\beta^2) \tag{6.5}$$

Condition (6.5) is satisfied for any angular velocity if the axis of symmetry is the greatest axis, otherwise at a fairly high angular velocity.

The equations of perturbed motion contain no second-order terms in the perturbations. Therefore, Theorem 2 cannot be used, and the question of the existence of the Lyapunov family in the case of (6.5) remains open. Note that, here, the interesting problem arises of the existence of a Lyapunov family in a degenerate but fairly typical case for mechanical systems.

In the case examined, the equations of perturbed motion reduce to the form (3.1). Here, in the function $Y_0(x)$ we have $m = 3$, and the coefficient $g = 0$. Lemma 2 cannot be used, and it is impossible to establish the existence of constant rotations in which the axis of the ellipsoid makes a non-zero angle with the vertical. In fact, there are no such rotations [18].

The conclusions obtained are summarized by the following theorem.

Theorem 5. Suppose a heavy, homogeneous ellipsoid performs, on an absolutely rough plane, permanent rotations about one of the axes (with semi-axis c) coinciding with the vertical, the angular velocity satisfying the condition $\omega^2 \neq 5g/(6c)$. Then, when rotation occurs about the smallest or middle axis or the greatest axis, but with a fairly high angular velocity $\omega > \omega^*$, such rotation always has a Lyapunov family of periodic motions. In the case of rotation about the smallest axis or the greatest axis with $\omega > \omega^*$, we have two such families, provided the non-resonance condition is satisfied.

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